Mathematics and Music Boxes

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Abstract

Music boxes which play a paper tape are fantastic tools for visually demonstrating some of the mathematical concepts in musical structure. The literal written notes in a piece can be transformed physically through reflections and rotations, and then easily played on the music box. Principles of topology can be demonstrated by playing loops and Möbius strips. Written music can also be transformed into different types of canons by sending it through multiple music boxes.

Introduction

Music boxes are familiar objects to most people, but most music boxes are manufactured to play only a single song. Some may be able to play a set of "records" manufactured for it, but even these are made specifically to play only the intended input in the intended way. Still, there is another kind of music box that plays a paper tape with holes punched in it [1]. Not only can the user write their own music by punching their own paper strips, but there is enough freedom in how the strip is played that the music can be transformed. Transformation is a basic concept in both music theory and geometry. Two musical phrases, or two triangles, can be congruent even if they look or sound different. These music boxes allow one to see notes as a set of points punched in paper, making both mathematics and music a visual, auditory, and tactile experience.

Related Work. Much has been written on the subject of music and mathematics, and the relationship between musical and mathematical transformations [2]. Others have also noted the mathematical potential of the music box, specifically with Möbius strips [3, 4].

Transforming a Piece. A blank strip of paper can be transformed four ways and still appear the same as it started. One could flip it vertically, flip it horizontally, or turn it upside-down, or of course leave it alone. Once it has holes punched in it, these transformations correspond to the transformations taught in every introductory music theory course: a motive (identity) can be found in inversion (reflection through horizontal axis), retrograde (reflection through vertical axis), or retrograde inversion (180° flip). Using a music box, these are the four ways one could insert the same strip of paper into the box.

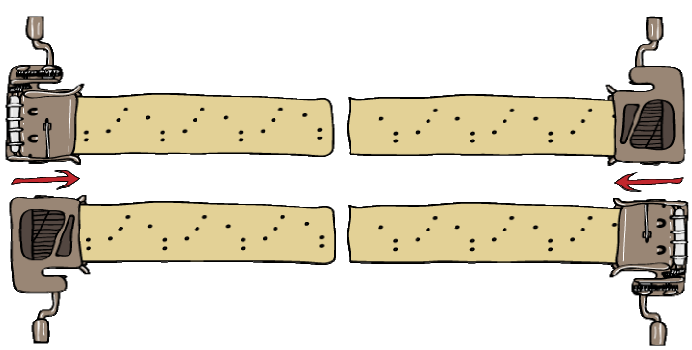

Figure 1: *Four ways to read the same strip of paper.*

Another way to think of these transformations is that the music, or points in space, are staying the same, but how we look at it is changing. One could think of the music box as a reader, scanning the strip in four different ways, as seen in figure 1.

The strip of paper can also be shifted forward or backwards in time, by playing it now or later. While this is not usually thought of as a transformation to a musician, it is known in geometry as a horizontal translation. If the strip is thinner than the width of the music box, the paper can also be translated vertically. This will make the pitch higher or lower, known to musicians as transposition. The two can be combined to play the music at both a different time and pitch, as seen in figure 2.

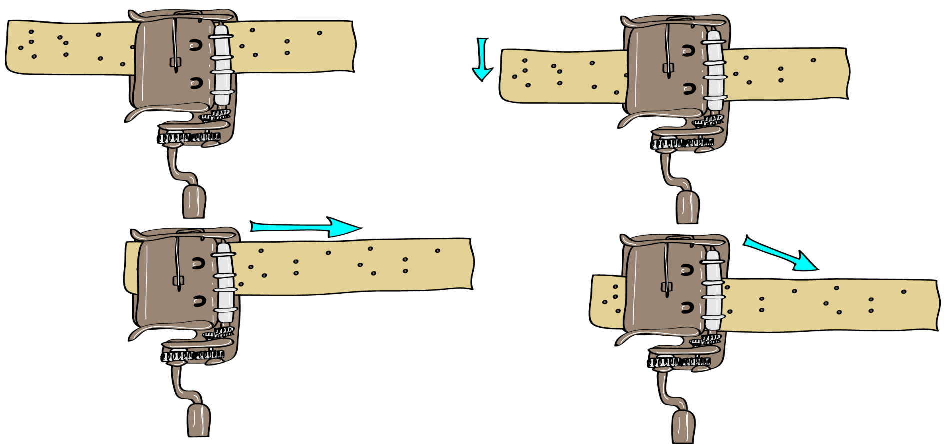

Figure 2: *Translating vertically, horizontally, or both.*

There is a difference between these musical transformations and transformations on the Euclidean plane. The musical plane has two distinct axes: time and pitch space. The only transformations allowed are ones that do not confuse the two axes. It would not make sense, for example, to try to put the strip through sideways (one could cut it down to a square and physically do it, of course, but that sort of transformation artificially sets a conversion rate between space and time). Because of this, some kinds of transformations have more than one version. A reflection along the vertical axis is different from a reflection along the horizontal axis, and playing a piece at a higher pitch is different from playing it tomorrow.

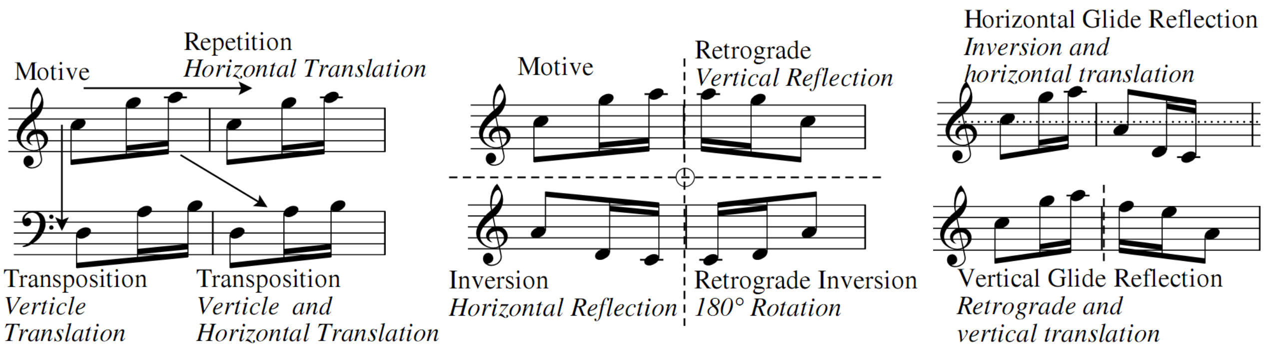

Figure 3: *Isometric Transformations in Musical Space*

The basic transformations of reflections and translations can be combined together to get glide reflections. All together, there are 8 distinct isometric musical transformations, as shown in figure 3 [5].

Loops. One can make the music repeat by physically taping the paper into a loop, after it has already been put through the music box. If you put a twist in the paper before taping it, you get a Möbius strip, as shown in figure 4. This gives you a pattern of the theme and its inversion. The music plays through once normally, and when it gets back to the beginning of the loop the paper will be taped on upside-down, and so the inversion will play. Putting two twists in the paper sounds the same as no twists, or indeed any even-number of twists. What matters is the topology of the loop: does it have one side or two? The music box only "scans" along one side, so if the paper has one side as a Möbius strip does, it will play both sides of the original un-twisted strip.

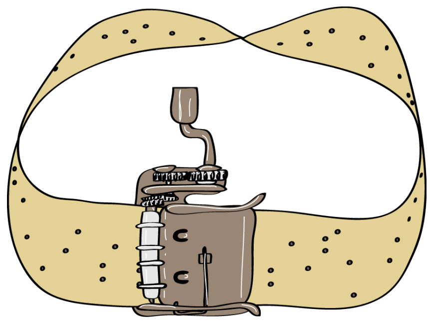

Figure 4: *Möbius Music Box*

The Möbius strip is a fun construction because you get twice the length of music out of your strip of paper, in a single repeating form. If you had a music box where you could turn the handle backwards to go backwards, you could start playing it backwards to get a pattern of the retrograde and retrograde inversion. In the case of the available box, once you tape your paper in a loop you are stuck going one way only (until you untape it).

The exception would be a theme that is, in music theory terms, "uninvertable," meaning that it sounds the same under inversion, and would therefore have a horizontal line of mirror symmetry. If this line of symmetry is at the same pitch that the music box inverts, the theme would sound the same when put through the box upside-down. If it weren't, the inversion would transpose the music.

A video demo of a Möbius music box is available on my website [6].

Canons. By putting a strip of paper through more than one music box, as seen in figure 5, one can play a canon. The same music is played twice, with a time shift on the overlap depending on how far apart the boxes are. One can also think of this as two scanners reading a strip at the same time.

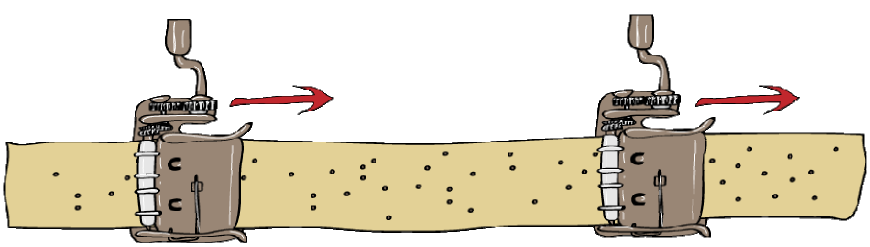

Figure 5: *Music Box Canon.*

Types of canons. A standard canon has only a shift in time between the two (or more) boxes, but there are many other types that use other transformations between the boxes. As shown in figure 6, one could also twist the piece of paper between the boxes to get another kind of canon, where the theme is followed by the inversion, or put a vertical shift between boxes to get a canon at the 5th (or any other interval in range). There are also types of canons where one voice plays the music forwards and the other in retrograde, for a crab canon, or upside-down and backwards, for a table canon, though unfortunately in this case the music boxes would run into each other and get stuck.

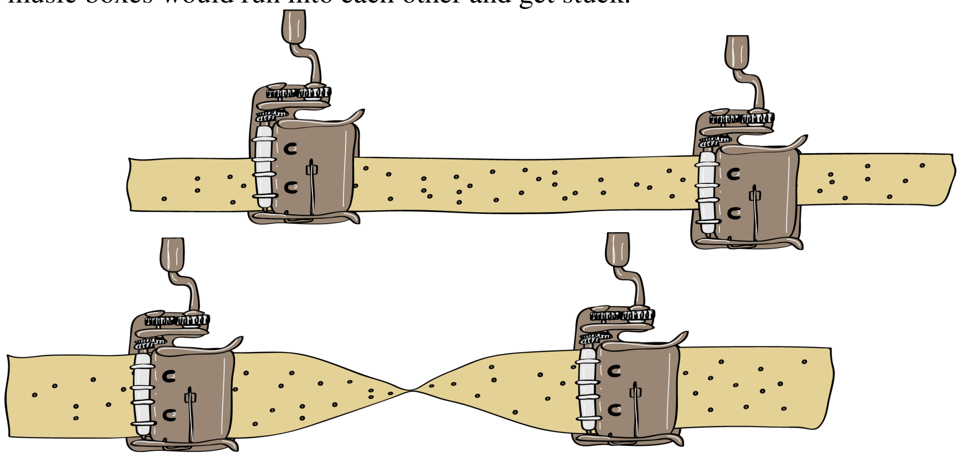

Figure 6: *More Kinds of Music Box Canons.*

A mensural canon is one where the voices play the same music at different tempos, which can be done by playing the boxes at different speeds. Some of these types of canons may not be as well known to non-musicians, but there are many examples by Bach, Mozart, and others.

Some of Bach's canons, notably in his *Musical Offering*, were presented as puzzles, with the music of the theme written out only once instead of showing what each instrument plays at each time. The music had clues as to which transformation would yield a good sounding piece, and it was up to the musicians to puzzle it out. Music boxes allow one to easily try out different combinations.

One of the best uses of music box canons is how it makes it easy for non-musicians to see and understand the structure of a canon. I demonstrate this in a video too, using a popular and easily recognized example: Pachelbel's Canon. Three music boxes are used to represent the three violins, and all three physically play the same strip of paper, one after another. The bass line, or basso continuo, is played by a fourth music box, playing a tight loop of paper that shows the repeating nature of the bass line. The form of the piece becomes easy to see [6].

References

[1] Kikkerland "DIY Mechanical Music Box Set:" http://www.kikkerlandshop.com/toys---games-music-boxes.html

[2] Wilfred Hodges, "The geometry of music," in *Music and Mathematics: From Pythagoras to Fractals*, Oxford University Press, 2003, pp. 91–111.

[3] Ranjit Bhatnagar's Möbius music box photo: http://www.flickr.com/photos/ranjit/3314000751/

[4] Roland Tremblay, *Inversus: Sankyo 20-Note Moebius Strip Plays Inverse Music*, Mechanical Music Digest, 2003. http://www.mmdigest.com/Sounds/Sankyo20/tremblay.html

[5] Vi Hart, "Symmetry and Transformations in the Musical Plane", in *Proceedings of the 12th Annual BRIDGES Conference: Mathematics, Music, Art, Architecture, Culture (BRIDGES 2009)*, Banff, Canada, July 2009.

[6] Vi Hart, *Music Boxes*: http://vihart.com/musicbox/