Mixing Mathematics and Music

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Abstract

Music and mathematics have been informing and inspiring each other for centuries. Composers and mathematicians have found many different ways to mix mathematics and music, and I highlight some different approaches to combining the two. What I am most interested in is balancing mathematics with the artistic process of composing. I present three recent projects which combine mathematical and musical creativity: using binary numbers in music, the organ function grinder, and the frieze pattern piece.

Introduction

We find great beauty in the way artists handle the limitations of their medium, and respect the skill it takes to work within constraints. For this reason, artists often impose constraints on their work. Creating beauty within constraints is a challenge for the artist, and we admire successful results. In this paper, I will limit discussion to the medium of music, and mathematics as a constraint.

Why mathematics and music? Connections between music and mathematics have been studied extensively by many throughout the literature; see, for example, [1, 2]. Though some argue that music and mathematics are inherently related, I choose to study them because I find both not only beautiful, but fun as well. Writing music with mathematical constraints is a genuinely enjoyable challenge, and listening to it can be a fun puzzle.

There are many ways of mixing mathematics and music, and I'd like to disambiguate between them. They generally fall into two main realms: analysis and composition.

In mathematical analysis of music, one uses mathematics to gain insight into preexisting music, which may or may not have been composed with mathematics in mind. One example is looking for mathematical features in music, such as finding the golden ratio in the form of a Mozart sonata, or studying the combinatorics of a set of notes in Bach [3].

Instead of looking for mathematics in music, one could use mathematics as a tool to automate classical analysis. The field of music information retrieval uses algorithms to try and extract features of music such as chords, beat, and genre [4].

As a composer, I am more interested in how mathematics can be used not to analyze existing music, but to create new music.

Many innovations in compositional techniques have appeared in the past century. Schoenberg used tone rows, John Cage created randomly generated compositions, and so on. Now, through the use of computers, we can hear a brand new piece played in real time. Computer algorithms have been designed to generate music, such as Wolfram Tones, which automatically generates a new piece of music according to certain rules [5]. But mathematical and algorithmic composition has been going on longer than computers have been around. For example, Mozart wrote a piece where one chooses from possible measures by rolling dice [6]. Though Mozart's rules are simpler than most computer music algorithms, the main concept is the same. The composition is not really any one piece it outputs, but the rules themselves.

Some music is composed freely but using ideas and inspiration from mathematics, which I call "mathematically inspired music." No mathematical rule is strictly adhered to, but mathematics influences some of the choices of the composer. An explanation of those choices can help the listener appreciate the music more, though generally it is impossible to discover the mathematical content without being told about it.

I prefer to have some compositional freedom in a single piece, while also creatively following strict mathematical rules. This style lies somewhere between algorithmic music and mathematically inspired music, the difference being that the listener can tell whether I am following the rules, but could not recreate the piece just by knowing the rules.

As with any art form, working within constraints can lead to new ideas. A creative work with a mathematical constraint must have a strict mathematical framework, but must also have human input. The composer must work creatively while following one or more rules.

In the next sections, I describe three of my recent projects that combine math and music in this last way.

Using Binary Numbers in Music

Rhythm is a binary feature of music, where 1 is 'play' and 0 is 'rest.' If each digit in a binary number represents one unit of time, a number can become a rhythm. Once a number or sequence of numbers is chosen, the rhythm of that line of music is entirely constrained. However, the composer has complete freedom to choose which pitches are played.

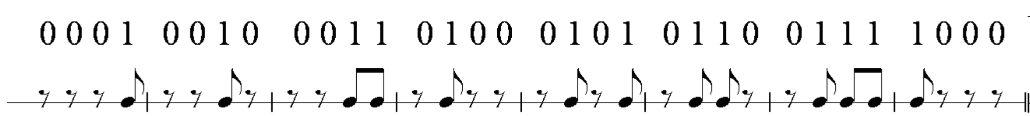

Figure 1: The numbers 1 through 8 in binary, and their rhythmic counterparts.

Choosing pitches within a constraint of rhythm is one element of creativity, but another is to choose which numbers to use. Some sequences of numbers have relationships that can be heard rhythmically, even if it is something as simple as the numbers getting larger, often causing there to be more 1s and so more notes to be heard.

For example, in the following example, the violin plays multiples of 1, the viola plays multiples of 2, the cello multiples of 3, and the bass of 5. Because the viola always plays a number twice as great as the violin, it has the same rhythm shifted over one.

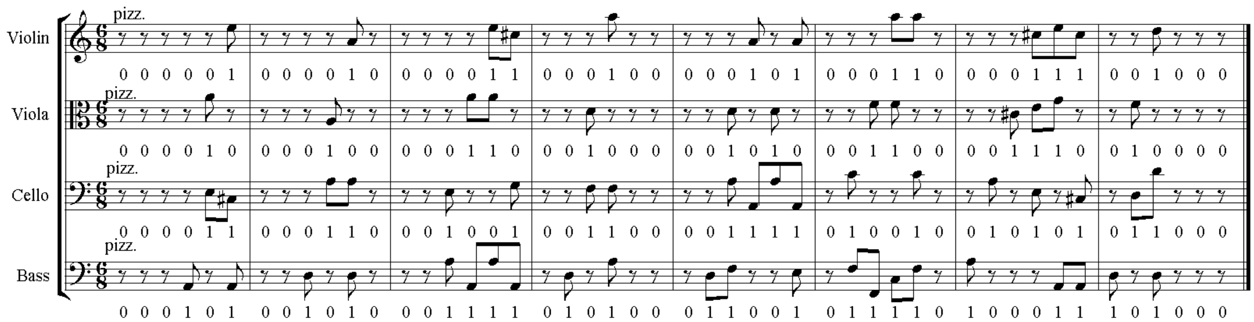

Figure 2: Multiples of 1, 2, 3, and 5

The shift between the rhythms of the violin and viola is a mathematical side-effect which an astute listener will notice, and it is easy to see in the score. The audible effect of the chosen numbers makes the existence of a constraint recognizable to the listener, and thus the piece becomes more interesting to listen to.

More examples and audio recordings can be found on my website[1], as well as a full paper on this topic [7].

Organ Function Grinder

Figure 3: The Organ Function Grinder at its debut

This is a project for the Math Midway, a museum exhibit aiming to make mathematics fun and accessible [8]. The user sets three dials to one of eight mathematical functions each, such as “add three,” “divide by two,” and “invert.” Then, they input a number. The number goes through the three functions in

[1] http://vihart.com/papers/binary

sequence, and then the machine outputs the result. My job was to write music that somehow goes along with this process.

Figure 4: Three operations act on an input number.

I decided to have the music mimic the mathematics. I composed a short melody for each of 24 possible input numbers, which plays once, then after each mathematical transformation it plays again, for a total of four times. For each mathematical transformation, I chose a musical transformation that I considered conceptually similar, so that the melody would transform along with the input number. Some seem obvious, such as “add three,” where I shift the melody up three semitones, “divide by two,” where the note values are divided in half, and “invert,” where the melody inverts around a specified pitch.

Figure 5: A melody and three transformations thereof.

Some mathematical functions don’t have as obvious a melodic counterpart. It does not immediately leap to mind how one would take the square root of a melody. An artistic choice must be made, so I chose to add a harmony below the melody for square root, and above for the square. The key also changes from major to minor or vice versa.

Figure 6: Nine, squared.

There are other artistic choices to be made as well. For example, multiplying a melody by two doesn’t make it actually go twice as slow, as doing this a few times would become boring to the listener. Instead, I scaled it in a way that lets you hear that it is going slower without making it literally twice as slow. Similarly, when inverting I chose to invert the melody but not the accompaniment, and I set the inversion point manually for each melody.

This may sound like algorithmic music, and certainly the output piece is, but I consider the composition to be the function rules and input melodies. While functions act algorithmically on the melody, so many artistic choices are made and so much of the piece is freely composed that no possible output would be unexpected to me.

The piece was extremely successful at its debut, with hundreds of children and adults choosing numbers and trying different functions. Many enjoyed the challenge of trying to get to a specific number, or to get an imaginary number, or even to divide by zero, to see what effect the music would produce. I implemented an algorithm that made imaginary numbers sound imaginary, and another that makes the organ grinder 'crash' if it has to divide by zero, which many people discovered through experimenting with their inputs.

Frieze Pattern Piece

Frieze patterns are patterns that repeat infinitely in one dimension. They can contain other symmetries besides repetition, symmetries which are often found in music: horizontal and vertical reflections, 180° rotations, and horizontal glide-reflections. For example, the *dizzy hop* has vertical mirrors (between each pair of feet) and points of rotation (between different pairs of feet).

Figure 7: Two dizzy hops, one in music and a visual counterpart above in footprints.

A study of all symmetries possible in music [9] inspired the composition of a seven-movement piece, where each movement contains one of the seven frieze patterns. I wrote one short motive that transforms throughout the seven movements to contain the proper symmetry. However, while I constrained myself to use strict symmetry in some of the piece, much of the piece uses the symmetry in a looser form, playing with variations and using the idea of the symmetry in creative ways.

The following excerpt from the second movement, "*Step,*" shows a motive alternating with its inversion (glide-reflection symmetry), but then step symmetry is broken as the piece moves forward. A note that doesn't follow a rule jumps out to the listener, so breaks in a pattern can be especially dramatic.

Figure 8: Excerpt with step symmetry from movement 2

Later in the same piece, step symmetry is seen in a looser way. Each instance of the motive is a glide-reflection of the previous instance, but the vertical distance grows shorter in a linear way. This causes the right and left "feet" to get closer together as they walk.

Figure 9: Excerpt with a variation on step symmetry

The rule of the movement (use strict step symmetry somewhere) is still being followed, but an additional and temporary rule is being followed for a short time. Because the main constraint doesn't need to be followed for the entire piece, adding variations such as this can augment it.

Much of the beauty in music arises from the tension between notes that follow the rule and notes that don't. It is common to hear patterns and repetition in music. Patterns and rules give us expectations as to what would happen next, if the pattern were followed. To break the rule, or alter it, can be dramatic and surprising. The effectiveness, of course, depends on how aware the listener is of the rule being followed.

Listeners are often intuitively aware of elements of music without realizing [10], so it is difficult to say which constraints can be heard and which cannot. Appreciation can certainly be gained through studying the piece, but the same can be said for all music. Constraints involving repetition, transposition or inversion have a musical history that is centuries old, so are likely to be understood on some level. However, even an obscure constraint can lead the composer to make choices she might not have otherwise made, which leads to interesting new music.

References

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